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Harmonic maps, Bäcklund–Darboux transformations and 'line solution' analogues

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Abstract

Harmonic maps from $\mathbb{R}^2$ or one-connected domain $\Omega \subset \mathbb{R}^2$ into $GL(m, \mathbb{C})$ and $U(m)$ are treated. The GBDT version of the Bäcklund–Darboux transformation is applied to the case of the harmonic maps and a new and simple algebraic procedure to construct new harmonic maps from the initial ones is given, using some methods from system theory. A new general formula on the GBDT transformations of the Sym–Tafel immersions is derived. A new class of the unitary harmonic maps with asymptotics along one line essentially different from the asymptotics in all other directions, similar in certain ways to line solutions, is obtained explicitly and studied.

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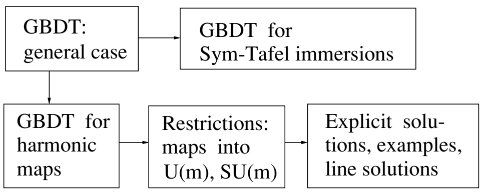
1. Introduction

Harmonic maps are actively studied in mathematical physics, differential geometry and soliton theory, and the famous Bäcklund–Darboux transformation can be fruitfully used for this purpose. Among various physical applications of harmonic maps are, in particular, harmonic map ansatz for skyrmions, isometric embeddings of the spacetime, problems of solid-state physics and important connections with the other integrable equations of mathematical physics (see [3, 6, 10, 31] and references therein). Since the original works of Bäcklund and Darboux various interesting versions of the Bäcklund–Darboux transformation have been introduced (see, for instance, [4, 5, 7, 8, 11–13, 26–28, 32]). In many of the references above, Bäcklund–Darboux transformations have been successfully applied to the studies of harmonic maps.

In our paper, we shall consider harmonic maps from $\mathbb{R}^2$ or one-connected domain $\Omega \subset \mathbb{R}^2$ into $GL(m, \mathbb{C})$ and $U(m)$. Here $GL(m, \mathbb{C})$ is the Lie group of $m \times m$ invertible matrices with the complex-valued entries, and $U(m)$ is its subgroup of unitary matrices. Correspondingly the map $u(x, y)$ $((x, y) \in \mathbb{R}^2)$ is called harmonic if it satisfies the Euler–Lagrange equation:

$$\overline{\partial}((\partial u)u^{-1}) + \partial((\overline{\partial}u)u^{-1}) = 0, \qquad \left(\partial = \frac{1}{2}\left(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\right), \overline{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right)\right). \tag{1.1}$$

We apply a version of the Bäcklund–Darboux transformation (so-called GBDT), developed in [15–20] and some other works of the author, to the case of the harmonic maps. Thus a new and simple algebraic procedure to construct new harmonic maps from the initial ones is given, using some methods from the system theory. As a result a new class of the unitary harmonic maps with asymptotics along one line essentially different from the asymptotics in all other directions, similar in certain ways to line solutions, is obtained explicitly and studied. In particular, the absolute value of the non-diagonal entry $(\widetilde{u})_{21}$ for $\widetilde{u}$, belonging to the family of the harmonic maps into $U(2)$ constructed in example 3.1, is non-decaying (even constant) along one line and decays exponentially in other directions. Compare with the characteristic properties of the line solutions discussed, for instance, in [29]. It is of interest that our approach allows us to include into consideration Darboux matrices with the high-order poles. (The case of a Darboux matrix with a second-order pole is studied in more detail in example 3.3.) A new formula on the GBDT transformations of the Sym–Tafel immersions is derived, which is true also for the much more general class of GBDT transformations, treated in [18]. The connections between our results and GBDT are presented in the following scheme:



GBDT version of the Bäcklund–Darboux transformation for harmonic maps is constructed in section 2. In particular, GBDT transformations of the Sym–Tafel immersions are given in proposition 2.8. Explicit solutions are treated in section 3.

2. GBDT version of the Bäcklund–Darboux transformation

Suppose u and its partial derivatives are continuously differentiable and u is a harmonic map into $GL(m, \mathbb{C})$. Then Euler–Lagrange equation (1.1) is equivalent [14] to the compatibility condition

$$\overline{\partial} G(x, y, \lambda) - \partial F(x, y, \lambda) + [G(x, y, \lambda), F(x, y, \lambda)] = 0, \qquad ([G, F] := GF - FG) \tag{2.1}$$

for system

$$\partial w(x, y, \lambda) = G(x, y, \lambda) w(x, y, \lambda), \qquad \overline{\partial} w(x, y, \lambda) = F(x, y, \lambda) w(x, y, \lambda), \tag{2.2}$$

where

$$G(x, y, \lambda) = -(\lambda - 1)^{-1} q(x, y), \qquad F(x, y, \lambda) = -(\lambda + 1)^{-1} Q(x, y), \tag{2.3}$$

$$q(x, y) = (\partial u(x, y)) u(x, y)^{-1}, \qquad Q(x, y) = -(\overline{\partial} u(x, y)) u(x, y)^{-1}. \tag{2.4}$$

Without loss of generality assume $(0, 0) \in \Omega$. To construct GBDT fix an integer $n > 0$ and five parameter matrices $A_1, A_2, S(0, 0), \Pi_1(0, 0)$ and $\Pi_2(0, 0)$, where A_1, A_2, S are $n \times n$ matrices, and Π_1, Π_2 are $n \times m$ matrices. We require also that $\pm 1 \notin \sigma(A_k)$ ($k = 1, 2$, σ—spectrum) and the identity

$$A_1 S(0,0) - S(0,0) A_2 = \Pi_1(0,0)\Pi_2(0,0)^* \tag{2.5}$$

holds. Now introduce matrix functions $\Pi_1(x,t)$ and $\Pi_2(x,t)$ using their values at $(x,y) = (0,0)$ and linear differential equations

$$\partial \Pi_1 = (A_1 - I_n)^{-1}\Pi_1 q, \qquad \overline{\partial}\Pi_1 = (A_1 + I_n)^{-1}\Pi_1 Q, \tag{2.6}$$

$$\partial \Pi_2^* = -q\Pi_2^*(A_2 - I_n)^{-1}, \qquad \overline{\partial}\Pi_2^* = -Q\Pi_2^*(A_2 + I_n)^{-1}, \tag{2.7}$$

where I_n is the $n \times n$ identity matrix. Similar to the case of the differentiation in real-valued arguments in [17, 18] the compatibility of both systems (2.6) and (2.7) follows from the compatibility condition (2.1).

Next, introduce $m \times m$ matrix function $S(x,y)$ via $S(0,0)$ and the partial derivatives:

$$\partial S = -(A_1 - I_n)^{-1}\Pi_1 q \Pi_2^*(A_2 - I_n)^{-1}, \qquad \overline{\partial} S = -(A_1 + I_n)^{-1}\Pi_1 Q \Pi_2^*(A_2 + I_n)^{-1}. \tag{2.8}$$

From formulae (2.1) and (2.6)–(2.8), it follows that $\overline{\partial}\partial S = \partial\overline{\partial} S$, i.e., $S_{xy} = S_{yx}$, where $S_x = \frac{\partial S}{\partial x} = \partial S + \overline{\partial} S$, $S_y = \frac{\partial S}{\partial y} = \mathrm{i}(\partial S - \overline{\partial} S)$. Thus the entries of S_x and S_y form potential fields and so equations (2.8) are compatible.

According to formulae (2.6)–(2.8) we have $\partial(A_1 S - S A_2) = \partial(\Pi_1\Pi_2^*)$ and $\overline{\partial}(A_1 S - S A_2) = \overline{\partial}(\Pi_1\Pi_2^*)$, which in view of the identity (2.5) implies the more general identity

$$A_1 S(x,y) - S(x,y) A_2 = \Pi_1(x,y)\Pi_2(x,y)^*. \tag{2.9}$$

Assume now that S is invertible and consider well known in the system theory transfer matrix function represented in the Lev Sakhnovich form [21–23]:

$$w_A(x,y,\lambda) = I_m - \Pi_2(x,y)^* S(x,y)^{-1}(A_1 - \lambda I_n)^{-1}\Pi_1(x,y). \tag{2.10}$$

The matrix function w_A is invertible [21]:

$$w_A(x,y,\lambda)^{-1} = I_m + \Pi_2(x,y)^*(A_2 - \lambda I_n)^{-1} S(x,y)^{-1}\Pi_1(x,y). \tag{2.11}$$

Formula (2.11) easily follows from the identity (2.9).

From the main theorem in [17, p 1253] it follows that w_A is the so-called Darboux matrix function.

Proposition 2.1. *Suppose $m \times m$ matrix function u and its partial derivatives are continuously differentiable and relations (1.1), (2.4) and (2.5)–(2.8) are valid. Then in the points of invertibility of S we have*

$$\partial w_A(x,y,\lambda) = \widetilde{G}(x,y,\lambda) w_A(x,y,\lambda) - w_A(x,y,\lambda) G(x,y,\lambda), \tag{2.12}$$

$$\overline{\partial} w_A(x,y,\lambda) = \widetilde{F}(x,y,\lambda) w_A(x,y,\lambda) - w_A(x,y,\lambda) F(x,y,\lambda), \tag{2.13}$$

where w_A is given in (2.10), G and F are defined using (2.3),

$$\widetilde{G}(x,y,\lambda) = -(\lambda - 1)^{-1}\widetilde{q}(x,y), \qquad \widetilde{F}(x,y,\lambda) = -(\lambda + 1)^{-1}\widetilde{Q}(x,y), \tag{2.14}$$

$$\widetilde{q}(x,y) = w_A(x,y,1) q(x,y) w_A(x,y,1)^{-1}, \tag{2.15}$$

$$\widetilde{Q}(x,y) = w_A(x,y,-1) Q(x,y) w_A(x,y,-1)^{-1}. \tag{2.16}$$

Suppose w is an $m \times m$ invertible matrix function satisfying formulae (2.2)–(2.4). According to equations (2.2) and formula (2.3) the equalities

$$\partial w(x,y,0) = q(x,y) w(x,y,0), \qquad \overline{\partial} w(x,y,0) = -Q(x,y) w(x,y,0) \tag{2.17}$$

hold. So in view of equalities (2.4) one can normalize w so that

$$w(x, y, 0) = u(x, y), \tag{2.18}$$

where u satisfies the Euler–Lagrange equation (1.1). Normalized in this way matrix function $w(x, y, \lambda)$ is called an extended (and corresponding to u) solution of the Euler–Lagrange equation or extended frame.

Theorem 2.2. *Suppose u satisfies the Euler–Lagrange equation (1.1), the conditions of proposition 2.1 are fulfilled, and w is an extended corresponding to u solution. Then the matrix function*

$$\widetilde{u}(x, y) := w_A(x, y, 0)u(x, y) \tag{2.19}$$

also satisfies the Euler–Lagrange equation. Moreover the matrix function

$$\widetilde{w}(x, y, \lambda) := w_A(x, y, \lambda)w(x, y, \lambda). \tag{2.20}$$

is an extended solution such that $\widetilde{w}(x, y, 0) = \widetilde{u}(x, y)$.

Proof. According to formulae (2.2) and (2.20) and to proposition 2.1 we have

$$\partial\widetilde{w}(x, y, \lambda) = \widetilde{G}(x, y, \lambda)\widetilde{w}(x, y, \lambda), \qquad \overline{\partial}\widetilde{w}(x, y, \lambda) = \widetilde{F}(x, y, \lambda)\widetilde{w}(x, y, \lambda). \tag{2.21}$$

From equations (2.21) it follows that the compatibility condition

$$\overline{\partial}\widetilde{G}(x, y, \lambda) - \partial\widetilde{F}(x, y, \lambda) + [\widetilde{G}(x, y, \lambda), \widetilde{F}(x, y, \lambda)] = 0 \tag{2.22}$$

is fulfilled. Condition (2.22) can be rewritten in the form

$$(\lambda + 1)\overline{\partial}\widetilde{q} - (\lambda - 1)\partial\widetilde{Q} + [\widetilde{Q}, \widetilde{q}] = 0,$$

i.e., the coefficients of the polynomial in λ on the left-hand side of the last equation turn to zero. In other words we have

$$\overline{\partial}\widetilde{q}(x, y) = \partial\widetilde{Q}(x, y) = -\tfrac{1}{2}[\widetilde{Q}(x, y), \widetilde{q}(x, y)]. \tag{2.23}$$

From formulae (2.18)–(2.20) it follows that $\widetilde{w}(x, y, 0) = \widetilde{u}(x, y)$. Therefore taking into account formulae (2.14) and (2.21) we obtain

$$\partial\widetilde{u}(x, y) = \widetilde{q}(x, y)\widetilde{u}(x, y), \qquad \overline{\partial}\widetilde{u}(x, y) = -\widetilde{Q}(x, y)\widetilde{u}(x, y). \tag{2.24}$$

Hence in view of formula (2.23) the matrix function $\widetilde{u}$ satisfies (1.1). Now we see that according to formulae (2.14), (2.21) and (2.24) $\widetilde{w}$ is an extended solution corresponding to $\widetilde{u}$. □

Theorem 2.2 presents a GBDT method to construct harmonic maps $\widetilde{u}$ into $GL(m, \mathbb{C})$ and corresponding extended solutions. According to formula (2.10) the choice of the eigenvalues of A_1 defines simple and multiple poles of the Darboux matrix w_A, and in this way our result is related to the interesting papers [2, 9, 30] on the pole data for the soliton solutions. Note that we do not require parameter matrix A_1 to be similar to diagonal (it may have an arbitrary Jordan structure).

Consider the case $uu^* = u^*u = I_m$, i.e., u is a harmonic map into $U(m)$. Then we have $u_xu^* + uu_x^* = u_yu^* + uu_y^* = 0$. Therefore from formula (2.4) it follows that

$$q^* = \tfrac{1}{2}\left(uu_x^* + \mathrm{i}uu_y^*\right) = -\tfrac{1}{2}\left(u_xu^* + \mathrm{i}u_yu^*\right) = Q. \tag{2.25}$$

Put now $A_1 = -A_2^* = A$ (i.e., assume $A_1 = -A_2^*$). Then, taking into account formulae (2.6), (2.7) and (2.25) we can assume $\Pi_2 = \Pi_1$ and denote Π_1 by Π. Further in this section we assume

$$A_1 = -A_2^* = A, \qquad \Pi_1(x, t) = \Pi_2(x, t) = \Pi(x, t), \quad \text{and} \quad S(0, 0) = S(0, 0)^*. \tag{2.26}$$

Hence, using formulae (2.8) and (2.25) we obtain $(\partial S)^* = \overline{\partial} S$, and so $S_x = S_x^*$, $S_y = S_y^*$. The last equality in formulae (2.26) now implies

$$S(x, y) = S(x, y)^*. \tag{2.27}$$

Corollary 2.3. *Suppose the equality $u^* = u^{-1}$, the conditions of theorem 2.2 and assumptions (2.26) are true. Then we have*

$$\widetilde{u}(x, y)^* = \widetilde{u}(x, y)^{-1}, \qquad \widetilde{q}(x, y)^* = \widetilde{Q}(x, y). \tag{2.28}$$

Proof. Identity (2.9) and definition (2.10) now take the form

$$AS(x, y) + S(x, y)A^* = \Pi(x, y)\Pi(x, y)^*, \tag{2.29}$$

$$w_A(x, y, \lambda) = I_m - \Pi(x, y)^* S(x, y)^{-1}(A - \lambda I_n)^{-1}\Pi(x, y). \tag{2.30}$$

Formula (2.11) takes the form

$$w_A(x, y, \lambda)^{-1} = I_m - \Pi(x, y)^*(A^* + \lambda I_n)^{-1} S(x, y)^{-1}\Pi(x, y). \tag{2.31}$$

In view of formulae (2.27), (2.30) and (2.31), we derive

$$w_A(x, y, \lambda)^{-1} = w_A(x, y, -\overline{\lambda})^*. \tag{2.32}$$

In particular, we have $w_A(x, y, 0)^{-1} = w_A(x, y, 0)^*$ and the first equality in formula (2.28) follows from the definition of $\widetilde{u}$. Moreover, according to formula (2.32) we have $w_A(x, y, -1)^{-1} = w_A(x, y, 1)^*$ and so the second equality in formulae (2.28) follows from formulae (2.15), (2.16) and (2.25). □

Remark 2.4. The uniton solutions have been introduced in the seminal paper [28]. Put

$$A_1 = aI_n, \qquad A_2 = bI_n, \qquad \pi = (a - b)^{-1}\Pi_2^* S^{-1}\Pi_1. \tag{2.33}$$

We shall derive some relations for π to compare with the limiting uniton case. According to (2.9) and (2.33) we easily obtain

$$\pi^2 = (a - b)^{-2}\left(\Pi_2^* S^{-1} A_1 \Pi_1 - \Pi_2^* A_2 S^{-1}\Pi_1\right) = \pi, \tag{2.34}$$

i.e., $\pi(x, y)$ is a projector. From (2.7) and (2.8) it follows that

$$\partial\left(\Pi_2^* S^{-1}\right) = -\widetilde{q}\,\Pi_2^* S^{-1}(A_2 - I_n)^{-1}. \tag{2.35}$$

Hence, taking into account (2.6) we have

$$\partial\left(\Pi_2^* S^{-1}\Pi_1\right) = -\widetilde{q}\,\Pi_2^* S^{-1}(A_2 - I_n)^{-1}\Pi_1 + \Pi_2^* S^{-1}(A_1 - I_n)^{-1}\Pi_1 q. \tag{2.36}$$

In view of (2.10), (2.11), (2.15) and (2.33), we obtain

$$\widetilde{q} = \left(I_m - \frac{a - b}{a - 1}\pi\right) q \left(I_m + \frac{a - b}{b - 1}\pi\right). \tag{2.37}$$

Using (2.33), (2.34) and (2.37) rewrite (2.36) as

$$\partial\pi = -(b - 1)^{-1}\frac{a - 1}{b - 1}\left(I_m - \frac{a - b}{a - 1}\pi\right) q\pi + (a - 1)^{-1}\pi q. \tag{2.38}$$

When $a = \overline{a}, b = -a$, we can assume $\Pi_1 = \Pi_2$, $S = S^*$, i.e., $\pi = \pi^*$ is an orthogonal projector and

$$\partial\pi = \frac{1 - a}{(a + 1)^2}\left(I_m - \frac{2a}{a - 1}\pi\right) q\pi + (a - 1)^{-1}\pi q. \tag{2.39}$$

The so-called singular Bäcklund transformation that transforms unitons into unitons deals with the case $a \to -1$. In this case, we have $(a+1)^{-1} \to \infty$, and so (2.39) implies $(I_m - \pi)q\pi = 0$ (see [28] and formula (5.130) [8]). Similar transformations are possible for the following equation:

$$\overline{\partial}\big(\Pi_2^* S^{-1}\Pi_1\big) = -\widetilde{Q}\Pi_2^* S^{-1}(A_2 + I_n)^{-1}\Pi_1 + \Pi_2^* S^{-1}(A_1 + I_n)^{-1}\Pi_1 Q.$$

It is of interest to consider harmonic maps into $SU(m)$. For that purpose introduce the notion of minimal realization. Any rational $m \times m$ matrix function φ that tends to D at infinity can be presented in the form

$$\varphi(\lambda) = D - C(A - \lambda I_r)^{-1}B, \tag{2.40}$$

where D is an $m \times m$ matrix, C is an $m \times r$ matrix, B is an $r \times m$ matrix, and A is an $r \times r$ matrix, $r \geqslant 0$, and the case $r = 0$ corresponds to $\varphi \equiv D$. This type representation is called a *realization* in system theory.

Definition 2.5. *Realization (2.40) is called minimal if the order r of A is the minimal possible. This order is called the McMillan degree of* φ.

Realization remains minimal under small perturbations of A, B, C.

Theorem 2.6. *Suppose the conditions of corollary 2.3 are fulfilled,* $\sigma(A) \cap \sigma(-A^*) = \emptyset$, *and realization (2.30) is minimal for some* $(x_0, y_0) \in \Omega$. *Then for each* $(x, y) \in \Omega$ *we have*

$$\det w_A(x, y, \lambda) = \prod_{k=1}^{n} \frac{\lambda + \overline{a_k}}{\lambda - a_k}, \tag{2.41}$$

where $\{a_k\}$ *are the eigenvalues of A taken with their algebraic multiplicity.*

Proof. It is well known that as $\sigma(A) \cap \sigma(-A^*) = \emptyset$ and (2.30) is a minimal realization of $w_A(x_0, y_0, \lambda)$, so $\det w_A(x_0, y_0, \lambda)$ has poles in all the eigenvalues of A and only there. Moreover, if $\det D \neq 0$, then the McMillan degrees of φ and φ^{-1} coincide. Thus (2.31) is a minimal realization of $w_A(x_0, y_0, \lambda)^{-1}$. Hence $\det w_A(x_0, y_0, \lambda)^{-1}$ has poles in all the eigenvalues of $-A^*$ and only there. Therefore, if all the eigenvalues of A are simple we obtain

$$\det w_A(x_0, y_0, \lambda) = \prod_{k=1}^{n} \frac{\lambda + \overline{a_k}}{\lambda - a_k}. \tag{2.42}$$

If A has multiple eigenvalues we prove (2.42) using small perturbations of A (and corresponding perturbations of S so that the identity (2.29) preserves). Finally, note that $\det w_A(x, y, \lambda) = \prod_{k=1}^{r} \frac{\lambda - \hat{a}_k}{\lambda - a_k}$ for each (x, y) with possible arbitrariness in the choice of the set of eigenvalues a_k of A and $\hat{a}_k$ of $-A^*$. Taking into account that w_A is also continuous we derive (2.41) from (2.42). □

The next corollary is immediate.

Corollary 2.7. *Suppose the conditions of theorem 2.6 are fulfilled,* $\det(\mathrm{i}A) \in \mathbb{R}$ *and* $u \in SU(m)$. *Then we have* $\widetilde{u} \in SU(m)$.

In the framework of his theory of 'soliton surfaces' A Sym associates with integrable nonlinear systems the corresponding λ-families of immersions

$$R(x, y, \lambda) = w(x, y, \lambda)^{-1}\frac{\partial}{\partial\lambda} w(x, y, \lambda), \tag{2.43}$$

where w are extended solutions [25]. On the other hand, our version of the Darboux matrix, which can be used in numerous important cases, including harmonic maps treated in this section, admits representation (2.10), where dependence on λ is restricted to the resolvent $(A_1-\lambda I_n)^{-1}$. Thus w_A is easily differentiated in λ. The next proposition expresses immersions generated by GBDT in terms of Π_1, Π_2 and S.

Proposition 2.8. *Suppose extended solution $\widetilde{w}$ is given by equality (2.20), where Darboux matrix w_A admits representation (2.10) and identity (2.9) holds. Then immersion $\widetilde{R} := \widetilde{w}^{-1}\frac{\partial}{\partial\lambda}\widetilde{w}$ admits representation*

$$\widetilde{R} = R - w^{-1}\Pi_2^*(A_2-\lambda I_n)^{-1}S^{-1}(A_1-\lambda I_n)^{-1}\Pi_1 w. \tag{2.44}$$

Proof. From (2.20) it easily follows that

$$\widetilde{R} = R + w^{-1}w_A^{-1}\left(\frac{\partial}{\partial\lambda}w_A\right)w. \tag{2.45}$$

According to (2.10) we have

$$\frac{\partial}{\partial\lambda}w_A(x,y,\lambda) = -\Pi_2(x,y)^*S(x,y)^{-1}(A_1-\lambda I_n)^{-2}\Pi_1(x,y). \tag{2.46}$$

Taking into account (2.9), (2.11) and (2.46) one obtains

$$\begin{aligned} w_A^{-1}\frac{\partial}{\partial\lambda}w_A &= \frac{\partial}{\partial\lambda}w_A - \Pi_2^*(A_2-\lambda I_n)^{-1}S^{-1}\Pi_1\Pi_2^*S^{-1}(A_1-\lambda I_n)^{-2}\Pi_1 \\ &= \frac{\partial}{\partial\lambda}w_A - \Pi_2^*(A_2-\lambda I_n)^{-1}S^{-1}((A_1-\lambda I_n)S - S(A_2-\lambda I_n))S^{-1}(A_1-\lambda I_n)^{-2}\Pi_1 \\ &= -\Pi_2^*(A_2-\lambda I_n)^{-1}S^{-1}(A_1-\lambda I_n)^{-1}\Pi_1. \end{aligned} \tag{2.47}$$

Substitute now (2.47) into (2.45) to obtain (2.44). □

Proposition 2.8 can be used to construct conformal CMC immersions.

3. Explicit solutions

For the simple seed solutions (see, for instance, solutions given by formulae (3.1) and (3.32)) equations (2.6) and (2.29) can usually be solved explicitly, so that equalities (2.19) and (2.30) provide us with the explicit expressions for harmonic maps. In this section, we consider several examples in greater detail. First similar to [8] we put $m=2$ and take the most simple seed solution of (1.1):

$$u(x,y) = \mathrm{e}^{(\tau\bar{z}-\bar{\tau}z)j} \in U(2), \qquad z = x+\mathrm{i}y, \qquad \tau \neq 0, \qquad j = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{3.1}$$

Transformations of this seed solution are treated in examples 3.1–3.3. Example 3.4 deals with the arbitrary m and seed solution of the form (3.32). It is of interest in examples 3.1 and 3.3 that asymptotics of the constructed maps differs in one particular direction and non-diagonal entries tend to zero, except, possibly, in this direction. (See relations (3.9), (3.11), (3.28) and (3.31).)

For u of the form (3.1) using (2.4) we obtain

$$q(x,y) = -\bar{\tau}j, \qquad Q(x,y) = -\tau j. \tag{3.2}$$

Partition the matrix function Π into columns

$$\Pi(x,y) = [\Phi_1(x,y) \quad \Phi_2(x,y)], \qquad \Pi(0,0) = [f_1 \quad f_2]. \tag{3.3}$$

Then according to (2.6), (2.26) and (3.2) we have

$$\Phi_1(x,y) = \exp(-\tau\overline{z}(A+I_n)^{-1} - \overline{\tau}z(A-I_n)^{-1})f_1, \tag{3.4}$$

$$\Phi_2(x,y) = \exp(\tau\overline{z}(A+I_n)^{-1} + \overline{\tau}z(A-I_n)^{-1})f_2. \tag{3.5}$$

Consider now the case $n = 1$, $A = a$ $(a \neq \pm 1, a \neq -\overline{a})$, $f_1, f_2 \neq 0$. Recall that the 1-soliton solution treated in [8, formulae (5.101) and (5.102)] had the following behaviour on the lines $z = x + \mathrm{i}y = \mu t$ $(\mu \in \mathbb{C}, \mu \neq 0, -\infty < t < \infty)$:

$$\widetilde{u} = gu, \qquad g(x,y) = c_\pm \,\mathrm{e}^{|kt|}(I_2 + o(1)), \qquad \text{for} \quad t \to \pm\infty, \quad c_\pm \in \mathbb{R}. \tag{3.6}$$

Our next example proves quite different.

Example 3.1. The GBDT transformation $\widetilde{u}$ of the seed solution is given via formula (2.19):

$$\widetilde{u}(x,y) = w_A(x,y,0)u(x,y),$$

where w_A is defined in (2.30). Let us study $\widetilde{u}$ on the lines $z = \mu t$. For Π on the right-hand side of (2.30), using (3.3)–(3.5) we obtain

$$\Pi(x,y) = [\mathrm{e}^{-bt} f_1 \quad \mathrm{e}^{bt} f_2], \qquad b := (a-1)^{-1}\overline{\tau}\mu + (a+1)^{-1}\tau\overline{\mu}. \tag{3.7}$$

Hence from (2.29) and (3.7) it follows that

$$S(x,y) = (a+\overline{a})^{-1}\Pi(x,y)\Pi(x,y)^* = (a+\overline{a})^{-1}(\mathrm{e}^{-(b+\overline{b})t}|f_1|^2 + \mathrm{e}^{(b+\overline{b})t}|f_2|^2). \tag{3.8}$$

Without loss of generality we can assume $b + \overline{b} \geqslant 0$. In view of (2.30), (3.7) and (3.8) for $b + \overline{b} > 0$ we have

$$\lim_{t\to\infty} w_A(x,y,0) = \begin{bmatrix} 1 & 0 \\ 0 & -\overline{a}/a \end{bmatrix}, \qquad \lim_{t\to-\infty} w_A(x,y,0) = \begin{bmatrix} -\overline{a}/a & 0 \\ 0 & 1 \end{bmatrix}. \tag{3.9}$$

The asymptotics differs on the line $\Gamma_0 = \{z : z = \mu t\}$, where $b + \overline{b} = 0$, i.e.,

$$\mu = \frac{\mathrm{i}\tau c}{(\overline{a}-1)(a+1)}, \qquad c = \overline{c} \neq 0. \tag{3.10}$$

Namely, on Γ_0 we have

$$w_A(x,y,0) = \frac{1}{a}\begin{bmatrix} \alpha & \beta(t) \\ \overline{\beta(t)}, & -\overline{\alpha} \end{bmatrix}, \qquad \alpha = \frac{a|f_2|^2 - \overline{a}|f_1|^2}{|f_1|^2 + |f_2|^2}, \tag{3.11}$$

$$\beta(t) = -\frac{(a+\overline{a})\overline{f_1}f_2}{|f_1|^2 + |f_2|^2}\,\mathrm{e}^{2bt}, \qquad (\overline{b} = -b). \tag{3.12}$$

Similar to the 'line solutions' studied, in particular, for KP (see, for instance, [1, 24, 29]) the asymptotics of our solution on Γ_0 essentially differs from asymptotics along other lines. Note also that only on Γ_0 the non-diagonal entries of $\widetilde{u}$ do not decay exponentially to zero (see figure 1 for the case $\tau = 2\mathrm{i}$, $a = 1 + \mathrm{i}$, $f_1 = 1$, $f_2 = 2\mathrm{i}$).

Example 3.2. This example deals with the case $n = 2$, $A = \mathrm{diag}\{a_1, a_2\}$, where diag means diagonal matrix, $a_k \neq \pm 1$, and $\sigma(A) \bigcap \sigma(-A^*) = \emptyset$. According to (3.3)–(3.5) on a line $z = \mu t$ we have

$$\begin{aligned} &\Pi(x,y) = [\mathrm{e}^{-Bt} f_1 \quad \mathrm{e}^{Bt} f_2], \qquad B = \mathrm{diag}\{b_1, b_2\}, \\ &b_k := (a_k - 1)^{-1}\overline{\tau}\mu + (a_k + 1)^{-1}\tau\overline{\mu}, \qquad (k = 1, 2). \end{aligned} \tag{3.13}$$

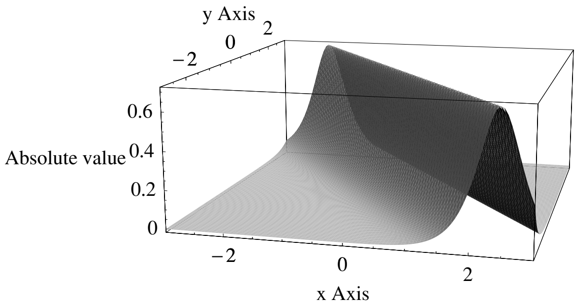


Figure 1. Asymptotics of $|\widetilde{u}_{21}(x, y)|$.

Supposing $b_1 + \overline{b_1} \neq 0$ we choose the sign of μ so that $b_1 + \overline{b_1} > 0$. Assume that $b_2 + \overline{b_2} > 0$ too (the case $b_2 + \overline{b_2} < 0$ can be treated similar). Let the entries f_{12} and f_{22} of f_2 be nonzero and let $a_1 \neq a_2$. Then in view of (2.29) and (3.13) we easily obtain

$$\Pi(x, y) = \begin{bmatrix} 0 & e^{b_1 t} f_{12} \\ 0, & e^{b_2 t} f_{22} \end{bmatrix} + o(1), \qquad t \to \infty, \tag{3.14}$$

$$S(x, y)^{-1} = \frac{1}{\det S(x, y)} \left(\begin{bmatrix} \frac{|f_{22}|^2}{a_2 + \overline{a_2}} e^{(b_2 + \overline{b_2})t} & -\frac{f_{12}\overline{f_{22}}}{a_1 + \overline{a_2}} e^{(b_1 + \overline{b_2})t} \\ -\frac{f_{22}\overline{f_{12}}}{a_2 + \overline{a_1}} e^{(b_2 + \overline{b_1})t} & \frac{|f_{12}|^2}{a_1 + \overline{a_1}} e^{(b_1 + \overline{b_1})t} \end{bmatrix} + o(1) \right), \tag{3.15}$$

$$\det S(x, y) = (1 + o(1)) \frac{|f_{12} f_{22}|^2 (|a_1|^2 + |a_2|^2 - a_1 \overline{a_2} - a_2 \overline{a_1})}{(a_1 + \overline{a_1})(a_2 + \overline{a_2})|a_1 + \overline{a_2}|^2} e^{(b_1 + \overline{b_1} + b_2 + \overline{b_2})t}. \tag{3.16}$$

After some calculations the asymptotics of $w_A(x, y, 0) = I_2 - \Pi^* S^{-1} A^{-1} \Pi$ follows from (3.14)–(3.16):

$$\lim_{t \to \infty} w_A(x, y, 0) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{\overline{a_1}\, \overline{a_2}}{a_1 a_2} \end{bmatrix}. \tag{3.17}$$

If in the last example, where A is diagonal, we put $a_1 = a_2 = a$ and $\det \Pi(0, 0) \neq 0$, we easily obtain a trivial answer $w_A(x, y, 0) = -(\overline{a}/a) I_2$. The case, where A is a Jordan box, is far more interesting.

Example 3.3. Suppose now that

$$n = 2, \qquad A = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}, \qquad a \neq \pm 1, \qquad a \neq -\overline{a}. \tag{3.18}$$

It follows that

$$\tau \overline{\mu} (A + I_2)^{-1} + \overline{\tau} \mu (A - I_2)^{-1} = b I_2 + c R, \qquad R = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \tag{3.19}$$

where b is given by the second relation in (3.7) and

$$c = -((a - 1)^{-2} \overline{\tau} \mu + (a + 1)^{-2} \tau \overline{\mu}). \tag{3.20}$$

According to (3.3)–(3.5) and (3.19) on a line $z = x + \mathrm{i}y = \mu t$ we have

$$\Pi(x, y) = [e^{-bt} (I_2 - ct R) f_1, \; e^{bt} (I_2 + ct R) f_2]. \tag{3.21}$$

In view of (3.18) identity (2.29) takes the form

$$(a+\overline{a})S+\begin{bmatrix} s_{21}+s_{12} & s_{22} \\ s_{22} & 0 \end{bmatrix}=\Pi\Pi^*, \tag{3.22}$$

where s_{kj} are the entries of S. From (3.21) and (3.22) it is immediate that

$$s_{22}(x,y)=(a+\overline{a})^{-1}(\mathrm{e}^{-(b+\overline{b})t}|f_{21}|^2+\mathrm{e}^{(b+\overline{b})t}|f_{22}|^2), \tag{3.23}$$

$$\begin{aligned} s_{12}(x,y)=\overline{s_{21}(x,y)}&=(a+\overline{a})^{-1} \\ &\times(\mathrm{e}^{-(b+\overline{b})t}(f_{11}-ctf_{21})\overline{f_{21}}+\mathrm{e}^{(b+\overline{b})t}(f_{12}+ctf_{22})\overline{f_{22}}-s_{22}(x,y)), \end{aligned} \tag{3.24}$$

$$s_{11}(x,y)=(a+\overline{a})^{-1}(\mathrm{e}^{-(b+\overline{b})t}|f_{11}-ctf_{21}|^2+\mathrm{e}^{(b+\overline{b})t}|f_{12}+ctf_{22}|^2-s_{12}(x,y)-s_{21}(x,y)). \tag{3.25}$$

Similar to example 3.1 assume $b+\overline{b}\geqslant 0$. Let $f_{21}\neq 0$ and $f_{22}\neq 0$ for simplicity. First we shall treat the lines, where $b+\overline{b}>0$. Taking into account (3.23)–(3.25) using standard calculations we derive

$$\det S(x,y)=(1+o(1))(a+\overline{a})^{-4}\,\mathrm{e}^{2(b+\overline{b})t}|f_{22}|^4. \tag{3.26}$$

One can easily see that

$$\Pi^*S^{-1}A^{-1}\Pi=a^{-2}(\det S)^{-1}\Pi^*\begin{bmatrix} s_{22} & -s_{12} \\ -s_{21} & s_{11} \end{bmatrix}\begin{bmatrix} a & -1 \\ 0 & a \end{bmatrix}\Pi. \tag{3.27}$$

From (3.23)–(3.27) it follows that

$$\begin{aligned} &\Pi^*S^{-1}A^{-1}\Pi\to\begin{bmatrix} 0 & 0 \\ 0 & a^{-2}(a^2-\overline{a}^2) \end{bmatrix}, \qquad \text{when} \quad t\to\infty, \quad i.e., \\ &\lim_{t\to\infty} w_A(x,y,0)=\begin{bmatrix} 1 & 0 \\ 0 & (a^{-1}\overline{a})^2 \end{bmatrix}, \qquad \text{for} \quad b+\overline{b}>0. \end{aligned} \tag{3.28}$$

As in example 3.1, the asymptotics of $w_A(x,y,0)$ differs on the line Γ_0, where $b+\overline{b}=0$. For this line using (3.23)–(3.25) we obtain

$$\det S=4(a+\overline{a})^{-2}|cf_{21}f_{22}|^2t^2+k_1t+k_0. \tag{3.29}$$

Taking also into account (3.27) we have

$$\Pi^*S^{-1}A^{-1}\Pi=4a^{-1}(a+\overline{a})^{-1}|cf_{21}f_{22}|^2(\det S)^{-1}\begin{bmatrix} t^2+O(t) & O(t) \\ O(t) & t^2+O(t) \end{bmatrix}. \tag{3.30}$$

Formulae (3.29) and (3.30) imply

$$\lim_{t\to\infty} w_A(x,y,0)=-(\overline{a}/a)I_2, \qquad \text{for} \quad b+\overline{b}=0. \tag{3.31}$$

Compare equalities (3.28) and (3.31). We would like to mention here that the extended solution $\widetilde{w}=w_Au$ has the pole of order two at $\lambda=a$:

$$\widetilde{w}(t,\lambda)=(a-\lambda)^{-2}(\det S(t))^{-1}\Pi(t)^*\begin{bmatrix} s_{22}(t) \\ -s_{21}(t) \end{bmatrix}[\mathrm{e}^{-bt}f_{21}\quad \mathrm{e}^{bt}f_{22}]+O\big((a-\lambda)^{-1}\big).$$

Our next example deals with an arbitrary m and a seed solution somewhat more general than that given in (3.1). Namely we put

$$u(x,y)=\mathrm{e}^{\overline{z}D-zD^*}\in U(m), \qquad D=\mathrm{diag}\{d_1,d_2,\ldots,d_m\}. \tag{3.32}$$

It follows that

$$q(x, y) = -D^*, \qquad Q(x, y) = -D. \tag{3.33}$$

Example 3.4. As in example 3.1, suppose $n = 1$, $A = a$ $(a \neq \pm 1, a \neq -\overline{a})$, $f_k \neq 0$. Here $1 \leqslant k \leqslant m$. Then, on a line $z = \mu t$ taking into account (2.6) and (2.29) we obtain

$$\Pi(x, y) = [\mathrm{e}^{-b_1 t} f_1 \quad \mathrm{e}^{-b_2 t} f_2 \quad \cdots], \qquad b_k := (a-1)^{-1}\overline{d_k}\mu + (a+1)^{-1} d_k \overline{\mu}, \tag{3.34}$$

$$S(x, y) = (a + \overline{a})^{-1} \sum_{k=1}^{m} |f_k|^2 \, \mathrm{e}^{-(b_k + \overline{b_k})t}. \tag{3.35}$$

In the generic situation there exists only one natural number k_+ and only one natural number k_- such that

$$\Re b_{k_+} = \min_{1 \leqslant k \leqslant m} \Re b_k, \qquad \Re b_{k_-} = \max_{1 \leqslant k \leqslant m} \Re b_k. \tag{3.36}$$

Then according to (3.34) and (3.36) we have

$$\lim_{t \to \infty} w_A(x, y, 0) = \mathrm{diag}\{1, \ldots, 1, -\overline{a}/a, 1, \ldots\}, \quad \text{where } -\overline{a}/a \text{ is the } k_+\text{th entry},$$

$$\lim_{t \to -\infty} w_A(x, y, 0) = \mathrm{diag}\{1, \ldots, 1, -\overline{a}/a, 1, \ldots\}, \quad \text{where } -\overline{a}/a \text{ is the } k_-\text{th entry}.$$

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